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LETTER TO THE EDITOR

Chaotic behaviour in sphaleron solution

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Abstract. We investigate the dynamical system derived from the $SU(2)$ Yang–Mills–Higgs theory with sphaleron solution for the case of spatial homogeneity. Numerical studies of the Poincaré surface of section and the Lyapunov characteristic exponents show that this system exhibits an order-to-chaos transition similar to one found in the $SU(2)$ magnetic monopole solution, which strongly suggests that such a transition is characteristic of the non-Abelian field theories with non-trivially topological solitons.

Recently there has been much interest in the sphaleron solution of the $SU(2)$ Yang–Mills–Higgs (YMH) equations with a doublet of scalar fields [1]. Its role becomes non-trivial in the presence of fermions and connects with a strong fermion number non-conservation.

The exact structure and dynamical properties of the sphaleron solution cannot be determined analytically. Thus, various properties of the sphaleron have been studied numerically by the variational techniques [2] and the application of the linear stability [3]. These numerical studies, however, have their limit for the study of the dynamical properties such as the instability and the bifurcation of the solution. Especially, the global structure of the sphaleron in the phase space has not been studied sufficiently and it is not clear whether or not the sphaleron exhibits a chaos. Since the chaos of the topological solitons, e.g. the magnetic monopole solution [4, 5], in the $SU(2)$ Yang–Mills (YM) and YMH theories has attracted much interest in various high-energy physics fields, it is important to examine the chaotic properties of the sphaleron solution.

In this letter we study the dynamical properties of the sphaleron solution from the viewpoint of chaos. In the present analyses we assume that the YM and the Higgs fields are spatially homogeneous. In other words, we consider the fields in the region of space in which their space fluctuation is small compared with their time fluctuations. This approximation has been widely used in the study of the dynamics of both the classical YM and YMH field theories [6]. From the homogeneous assumption the system we consider is reduced to a nonlinear mechanical system.

The sphaleron solution is a saddle-point configuration in the $SU(2)$ YMH theory, whose Lagrangian density is given by [1]

$$L = -\tfrac{1}{4}F^a_{\mu\nu}F^{\mu\nu a} + (D_\mu\phi)^\dagger(D^\mu\phi) - V(\phi) \qquad (1)$$

where $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\varepsilon^{abc}A^b_\mu A^c_\nu$, $(D_\mu\phi) = \partial_\mu\phi - \mathrm{i}gA_\mu\phi$, and $A_\mu = A^a_\mu T^a$. The element of the gauge group is given by $T^a = \sigma^a/2$. Here σ^as denote the Pauli matrices. The Higgs potential is $V(\phi) = \lambda(\phi^\dagger\phi - \tfrac{1}{2}v^2)^2$.

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The Euler–Lagrange equations are

$$(D_\nu F^{\mu\nu})_a = -\mathrm{i}g(\phi^\dagger T^a D_\mu\phi - (D_\mu\phi)^\dagger T^a\phi) \tag{2}$$
$$(D^\mu D_\mu\phi) = -2\lambda(\phi^\dagger\phi - \tfrac{1}{2}v^2)\phi\,. \tag{3}$$

From the homogeneous approximation, i.e. $\partial_i A_i^a = 0$ and $\partial_i\phi = 0$, and the gauge condition as $A_0^a = 0$, the above field equations reduce to the following equations of motion:

$$\ddot{A}_i^a + g^2(A_i^a A_j^b - A_j^a A_i^b)A_j^b + \tfrac{1}{4}g^2 A_i^b\phi^\dagger\{\sigma^a,\sigma^b\}\phi = 0 \tag{4}$$
$$\ddot{\phi} + \tfrac{1}{4}g^2 A_i^a A_i^b\sigma^a\sigma^b\phi + 2\lambda(\phi^\dagger\phi - \tfrac{1}{2}v^2)\phi = 0 \tag{5}$$

and the Gauss-law constraint is

$$\varepsilon_{abc}A_i^b\dot{A}_i^c - \frac{\mathrm{i}}{2}(\phi^\dagger\sigma^a\dot{\phi} - \dot{\phi}^\dagger\sigma^a\phi) = 0\,. \tag{6}$$

Now we take the following ansatz as $A_i^a = \varepsilon_{aij}a_j(t)$ and $\phi = \mathrm{i}\sigma\cdot b(t)\,\binom{0}{1}$ for the YM field and the doublet of the Higgs scalar fields, respectively. In order to fulfill the Gauss-law constraint, we put $a_i(t) = x(t)$ and $b_i(t) = y(t)$, $i = 1, 2, 3$. After appropriately rescaling the variables, we obtain the system of two degrees of freedom as

$$\ddot{x} = -\tfrac{1}{2}x^3 - \tfrac{1}{4}y^2x \tag{7}$$
$$\ddot{y} = -\tfrac{1}{4}x^2y - \kappa(y^2-1)y \tag{8}$$

where $\kappa = \lambda/g^2$ is the coupling constant. The resulting system has the Hamiltonian as

$$H = \tfrac{1}{2}(p_x^2 + p_y^2) + W(x, y) \tag{9}$$
$$W(x, y) = \tfrac{1}{8}x^4 + \tfrac{1}{8}x^2y^2 + \tfrac{1}{4}\kappa(y^2-1)^2 \tag{10}$$

which describes the motion of a particle in a two-dimensional potential well $W(x, y)$.

We first apply the Poincaré surface-of-section method [7] to the system of equations (9) and (10). This section is defined by $x = 0$, $p_x \geqslant 0$, which is a projection on the (y, p_y) plane. Regular regions on the surface of the section are characterized by sets of invariant Kolmogorov–Arnold–Moser (KAM) curves, whereas irregular regions are characterized by a scatter of points limited to a finite phase space due to the energy conservation. We have numerically integrated (7) and (8) with a fourth-order Runge–Kutta routine with a time step, Δt, equal to 10^{-2}. The size of Δt is chosen so that any reduction of the size does not cause significant change in the results. Figure 1 depicts the surfaces of section with the energy $E = 1.0$ and the coupling constant $\kappa = 0.1$, 0.4, and 1.0. We see from figure 1(a) that the entire phase space is almost covered by the KAM curves and thus the system is near-integrable. As we increase κ, some of the KAM curves begin to break up and we observe the coexistence of the KAM curves and stochastic orbits as shown in figure 1(b). As we further increase κ, the KAM curves and islands disappear and almost all regions of the phase space turn into the stochastic region as shown in figure 1(c). Thus we see that the relative area of the stochastic region strongly depends on κ and there is a transition from an almost all-regular state to an almost all-irregular state. We have made similar analyses for $E = 10.0$ and $E = 100.0$ and confirmed that the above results do not change qualitatively.

In order to determine whether or not our system exhibits chaos, we need to evaluate the Lyapunov characteristic exponents [7]. They are related to the exponentially fast divergence

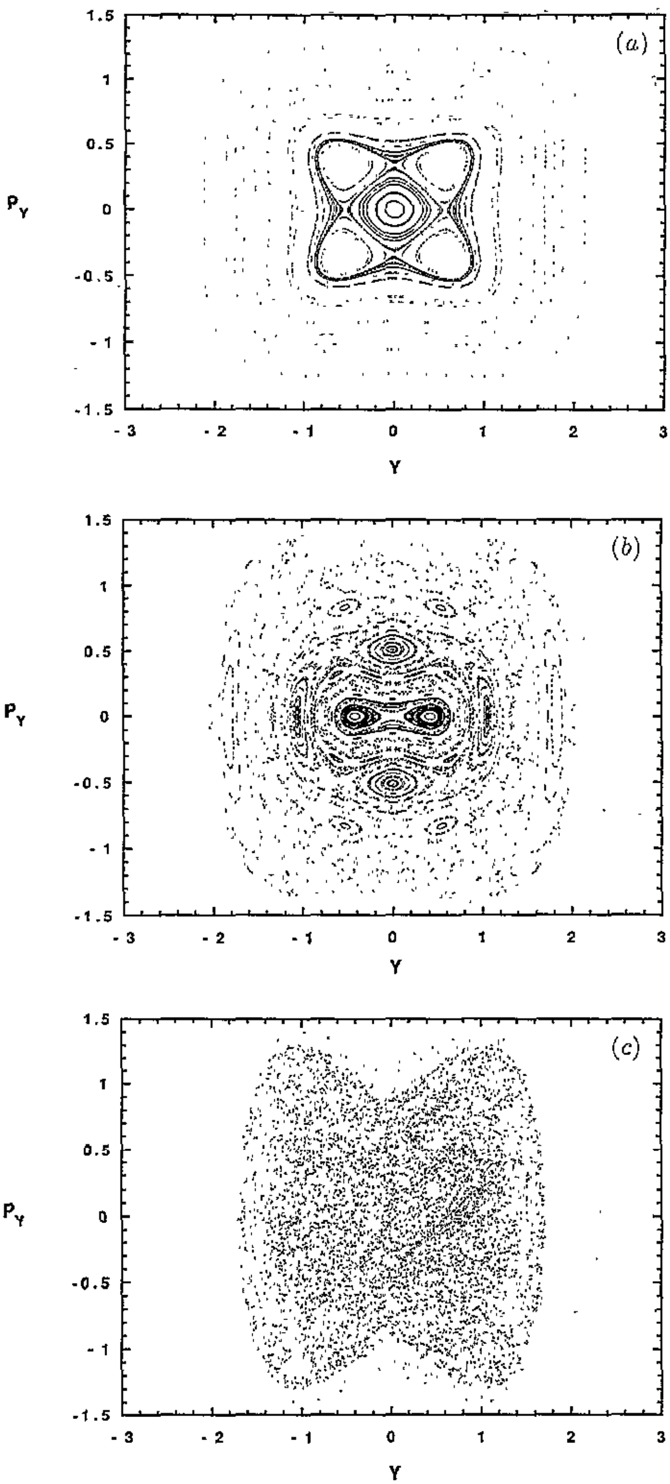

Figure 1. The Poincaré surface of section with $E = 1.0$. (*a*) $\kappa = 0.1$, (*b*) $\kappa = 0.4$, (*c*) $\kappa = 1.0$.

or convergence of nearby orbits in the phase space. The spectrum of the Lyapunov exponents is defined by the long-term evolution of an infinitesimal M-sphere of initial conditions in the M-dimensional phase space [7, 8]. During the evolution the sphere is becoming distorted into some ellipsoidal shape. The Lyapunov exponents σ_{L_i} ($i = 1, 2, \ldots, M$) measure the exponential growth of the principal axes of this ellipsoid as follows:

$$\sigma_{L_i} = \lim_{t\to\infty} \frac{1}{t} \ln \frac{p_i(t)}{p_i(0)} \tag{11}$$

where $p_i(t)$ is the length of ith ellipsoidal principal axis and σ_{L_i} are ordered from the largest to the smallest.

A chaotic system is established when some of σ_{L_i} are positive while a regular one *is established when $\sigma_{L_i} = 0$. The value of σ_{L_1} measures the largest of these exponential* growths. Figure 2 shows the result of σ_{L_1} versus time, t, for $E = 1.0$ calculated for six orbits. We see an apparent convergence of $\sigma_{L_1} \simeq 0.1$ for three of the orbits which are taken with initial conditions in the stochastic regions in figure 1(b). On the other hand, the convergence of $\sigma_{L_1} \simeq 0$ comes from the other three orbits which have initial conditions on the KAM curves in figure 1(b). This result shows that the chaotic and ordered regions coexist in figure 1(b) and indicates the onset of the chaos [7, 8], Thus we can see that this system exhibits an order-to-chaos transition similar to the one found in the magnetic monopole solution [9]. It is worthwhile pointing out that the Lyapunov exponent σ_{L_1} corresponding to the regular orbits has a power-law dependence on time, which reflects the typical behaviour of the long-time correlation in the Hamiltonian system. Indeed, our results in figure 2 have the same tendency as one obtained in the Hénon–Heiles problem [8].

In order to study how the relative measure of the chaotic region depends on κ, we calculate σ_{L_i} for many initial points in the following manner. Initial points are chosen on a grid of the (x, y) space subdivided into 50×50 bins, and 10^6 successive iterations of each initial point are computed. To judge the convergence of σ_{L_1}, we used the results of the numerical simulation done in figures 1 and 2. They indicate that the value of σ_{L_1} less than

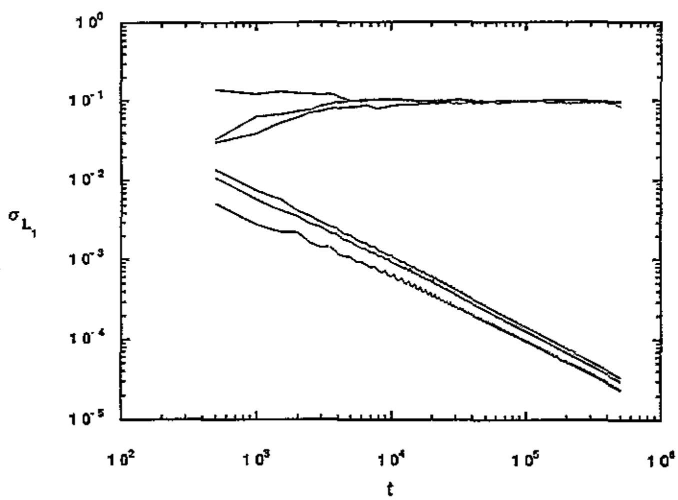

Figure 2. Behaviour of σ_{L_1} at $E = 1.0$ and $\kappa = 0.4$ for six initial points taken in the stochastic (three curves approaching to 0.1) or regular (other three curves) regions in figure 1(b).

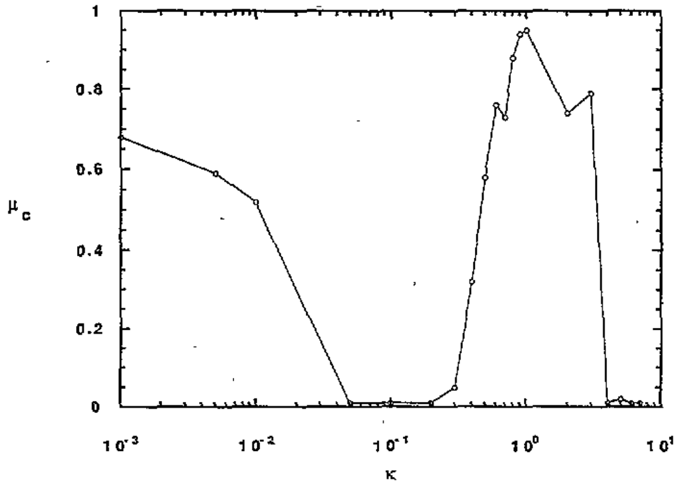

Figure 3. Fraction μ_c of the chaotic region in the phase space versus the coupling constant κ at $E = 1.0$.

$\sigma^* \approx 0.001$ approaches zero. Thus we regard an initial point (x_0, y_0) as regular when this (x_0, y_0) leads to σ_{L_1} less than σ^*. From these criteria we determined the fraction μ_c of bins occupied by the initial points which lead to chaos ($\sigma_{L_1} > \sigma^*$). Figure 3 shows the result for $E = 1.0$, which indicates that μ_c is almost zero for $0.05 \leqslant \kappa \leqslant 0.20$ and $\kappa \geqslant 4.00$. Especially, it is shown that the system becomes near-integrable for $\kappa \geqslant 4E$, at which point the topology of the potential well of (10) changes. Qualitatively the same tendency was also obtained for $E = 10.0$ and $E = 100.0$.

In conclusion, we have studied the chaotic properties of the sphaleron solution in the $SU(2)$ YMH theory in the limit of spatially homogeneous fields. From the results on the Poincaré surface of section and the Lyapunov characteristic exponents, we have found that the sphaleron solution exhibits the order-to-chaos transition, whose threshold depends on the coupling constant. Such a transition phenomenon has also been observed in the case of the magnetic monopole solution of the $SU(2)$ YMH theory [9].

Let us, finally, comment on our results. First, we point out the reason for the appearance of similar transition in both solutions by noticing the difference of the representation of the ϕ fields. In the sphaleron case ϕ belongs to the fundamental representation while in the monopole case ϕ to the adjoint representation, $(T^a)_{ij} = \mathrm{i}\varepsilon_{aij}$. These differences reflect only the strength of the coefficient x^2y^2 in (10). The integrability of this Hamiltonian can be examined by the Painlevé analysis [10]. Its integrability condition to (10) holds only special values of the coefficients $(c_1, c_2) = (c_1, 0)$ or (κ, κ) in c_1x^4 and $c_2x^2y^2$, which cannot be satisfied in the present system. Thus the system we considered is non-integrable so that it seems to be natural that the similar order-to-chaos transition is observed for both topological soliton solutions. Since our dynamical systems obtained for homogeneous fields are finite-dimensional ones, they are an extreme simplification of a full infinite-dimensional field theory. Concerning the chaos, however, it has been shown in the case of the magnetic monopole solution that consistent results can be reproduced both in the field theories [5] and their homogeneous versions [9]. Thus it seems that the order-to-chaos transition is characteristic of the non-Abelian field theories with non-trivially topological solitons such as the monopole and the sphaleron.

Second, the role of chaos in the field theories has been discussed for several issues in high-energy physics. For instance, it has been speculated that the Kolmogorov–Sinai entropy is responsible for the mechanism of the entropy production associated with the quark pair production in the quark–gluon plasma [11]. The chaos of the magnetic monopole solutions has been linked to the problem of the colour confinement [12]. In contrast with the monopole solution, however, it seems that the chaos of the sphaleron solution has not attracted much attention in the study of the sphaleron transition [13]. Especially, in the numerical study of such a transition the chaotic properties of the solution become significant in the course of solving the classical equations of motion in real time. Thus it would be of importance to investigate how the existence of the order-to-chaos transition affects the transition rate since the fluctuation causes the classical transitions from one vacuum to another passing over the sphaleron.

Third, we note the case of the Abelian gauge theories with the topological solitons. This case leads to a very different form of the potential from the non-Abelian case because of the lack of the non-Abelian term proportional to x^4 in (10). Since the potential well in this case is open to the x-axis direction, the motion of a particle with an energy E is not bounded by the line with a potential energy $W = E$. In these circumstances the orbit of a particle in the well is apt to move stochastically [14] so that we would have the chance to encounter a quite new pattern of the onset of chaos in the Abelian case. Thus it is interesting to investigate the dynamical property of the $U(1)$ vortex solution in the Abelian Higgs model [15] from the viewpoint of chaos.

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